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Aggregation–fragmentation processes and decaying three-wave turbulence

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We use a formal correspondence between the isotropic three-wave kinetic equation and the rate equations for a nonlinear fragmentation–aggregation process to study the wave frequency power spectrum of decaying three-wave turbulence in the infinite capacity regime. We show that the transient spectral exponent is $\lambda+1$, where λ is the degree of homogeneity of the wave interaction kernel and derive a formula for the decay amplitude. When $\lambda=0$ the transient exponent coincides with the thermodynamic equilibrium exponent leading to logarithmic corrections to scaling which we calculate explicitly for the case of constant interaction kernel.

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Wave turbulence is a theory of the statistical evolution of ensembles of weakly nonlinear dispersive waves. It has been applied to capillary waves on fluid interfaces, gravity waves on the ocean, acoustic turbulence, and various limits of plasma and geophysical turbulence. For a review of the theory and applications see [1]. The key feature is the fact that weak nonlinearity permits the consistent derivation [2] of a wave kinetic equation describing the time evolution of the frequency power spectrum, $N_\omega(t)$. When sources and sinks of energy, widely separated in frequency, are added to the wave kinetic equation, it can be shown to have exact stationary solutions corresponding to a cascade of energy through frequency space from the source to the sink. The cascade solution is known as the Kolmogorov-Zakharov (K-Z) spectrum; it describes an intrinsically nonequilibrium state of the wave field. A lot is known about the stationary K-Z spectra, their scaling exponents, locality, and stability. Time-dependent solutions of the wave kinetic equation are much less explored. A scaling theory of the development of the stationary state in the case of forced wave turbulence was initiated in [3] and developed further in [4]. Numerical investigations [5,6] have suggested, however, that there are unexplained dynamical scaling anomalies in many cases. Aside from the detailed analysis of gravity waves [7,8], there are very few general results known about time-dependent solutions which describe decaying turbulence where the spectrum decays in the absence of external forcing.

The subject of aggregation–fragmentation kinetics, having its origins in physical chemistry, has—at first sight—rather little to do with waves or turbulence. This field concerns itself with ensembles of particles which aggregate or fragment upon contact. The principal quantity of interest is $n_i(t)$, the density of clusters of mass i at time t. It satisfies a kinetic equation, a Smoluchowski coagulation equation, in the case of pure aggregation. If clusters also break up, additional terms should be added to the Smoluchowski equation. See [9] for a review of rate equations describing aggregation–fragmentation kinetics. In this field, in strong contrast with wave kinetics, almost all theoretical effort has historically been focused on determining the time evolution of $n_i(t)$ from the underlying kinetic equation. As a result, a comprehensive scaling theory of the solutions of the Smoluchowski equation has been constructed (see [10] for a review). Although there is a conceptual analogy [11] between energy transfer between scales in turbulence and mass transfer between clusters in aggregation, it is only recently that this analogy has been made quantitatively useful. Concepts from turbulence have proven useful in analyzing certain aggregation problems [12–14]. Furthermore, the kinetic equation describing isotropic three-wave turbulence can be rewritten as a set of rate equations for an aggregation–fragmentation process with an unusual nonlinear fragmentation mechanism [15]. This correspondence opens the door for the application of ideas and techniques from aggregation–fragmentation kinetics to wave turbulence and suggests new problems within aggregation–fragmentation kinetics.

Wave resonances lead to forward transfer of energy between frequencies, which looks like an aggregation process: $(i)\oplus(j)\to(i+j)$. Backscatter of energy leads to an unusual fragmentation process, $(i)\oplus(i+j)\to(i)\oplus(i)\oplus(j)$. It is nonlinear while typically [16] the fragmentation mechanism is linear: $(i+j)\to(i)\oplus(j)$. Nonlinear collision-controlled fragmentation has been studied in the past (see [17] and references therein) with rules which are somewhat similar to the above rule, but here only the larger particle breaks up in a process similar to fission. The resulting kinetic equation is

$$\frac{\partial N_{\omega_1}}{\partial t} = S_1[N_\omega] + S_2[N_\omega] + S_3[N_\omega]. \tag{1}$$

The first "collision integral,"

$$\begin{aligned} S_1[N_\omega] = &\int K(\omega_3,\omega_2) N_{\omega_2} N_{\omega_3} \delta(\omega_1-\omega_2-\omega_3)\, d\omega_2 d\omega_3 \\ &- \int K(\omega_3,\omega_1) N_{\omega_1} N_{\omega_3} \delta(\omega_2-\omega_3-\omega_1)\, d\omega_2 d\omega_3 \\ &- \int K(\omega_1,\omega_2) N_{\omega_1} N_{\omega_2} \delta(\omega_3-\omega_1-\omega_2)\, d\omega_2 d\omega_3, \end{aligned}$$

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is mathematically equivalent to aggregation [10]; the next two, written explicitly in [15,18], correspond to fragmentation. $K(\omega_1,\omega_2)$ is the wave interaction kernel which encodes the details of the specific physical problem under consideration. It is taken to be a homogeneous function of degree λ. Among the common examples of three-wave turbulence, capillary waves have $\lambda=2$, acoustic turbulence has $\lambda=2$, and quasi-two-dimensional Alfvèn waves have $\lambda=3$. The detailed form of $K(\omega_1,\omega_2)$ arising in these applications is often very complicated. Many questions, however, can be answered from the value of λ only. We therefore work mostly with model kernels which allow us to exhibit general structural features without the algebraic complexity necessary to treat particular examples.

We assume the *scaling hypothesis*: there exists a typical scale $s(t)$, such that $N_\omega(t)$ is asymptotically of the form

$$N_\omega(t) \sim s(t)^{-a}F(\xi), \quad \xi=\frac{\omega}{s(t)}. \tag{2}$$

It then follows from Eq. (1) that the typical frequency $s(t)$ and the scaling function $F(\xi)$ satisfy the equations

$$\frac{ds}{dt}=s^{\lambda-a+2}, \tag{3}$$

$$-aF-\xi\frac{dF}{d\xi}=S_1[F(\xi)]+S_2[F(\xi)]+S_3[F(\xi)]. \tag{4}$$

The scale $s(t)$ is defined as a ratio of moments: $s(t)=M_2(t)/M_1(t)$, where $M_n(t)=\int_0^\infty w^n N_\omega(t)d\omega$. Often the scaling function $F(x)$ diverges at small ξ,

$$F(\xi)\sim A\xi^{-x} \quad \text{as} \quad \xi\to 0. \tag{5}$$

The exponent x is the transient spectral exponent or the polydispersity exponent. The shape of $N_\omega(t)$ for large time is determined by the small ξ behavior of $F(\xi)$. In aggregation problems this divergence is often encountered and the exponent x can be difficult to determine [10,19]; in some seemingly simple models the exponent x remains unknown. An important lesson from this work is that one should be particularly careful when $x\geq 1$.

We restrict ourselves to values of the homogeneity index λ in the range $0\leq\lambda<1$, ensuring that the cascade has finite capacity so there is no dissipative anomaly or gelation transition. The case $\lambda>1$ is much trickier and is explored numerically in [20]. For finite capacity systems energy is conserved for all time by the wave interactions. We focus here on decaying turbulence. (The forced case is more straightforward and will be discussed elsewhere [18].) In the absence of a dissipative anomaly, the initial energy is conserved, $M_1(t)=1$. This, together with the scaling ansatz [Eq. (2)], determines the exponent a,

$$s^{2-a}\int_0^\infty \xi F(\xi)d\xi=1, \tag{6}$$

so that $a=2$. We then expect the scaling

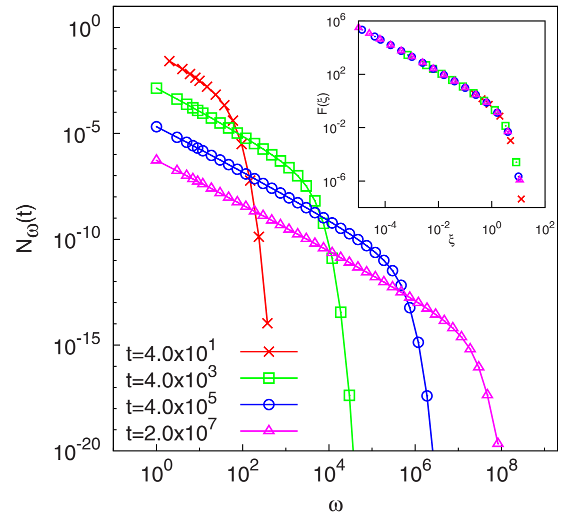

FIG. 1. (Color online) Time evolution of wave spectrum in the decay case. The main panel shows snapshots of $N_\omega(t)$ at a succession of times. The inset shows the data collapsed according Eq. (7).

$$N_\omega(t)\sim s^{-2}F(\omega/s) \quad \text{with} \quad s\sim s_0 t^{1/(1-\lambda)}. \tag{7}$$

Figure 1 presents numerical simulations of the decay of a monochromatic initial spectrum, $N_\omega(0)=\delta_{\omega,1}$, for constant wave interaction kernel, $K(\omega_1,\omega_2)=1$ $(\lambda=0)$. The data collapse in the inset verifies the scaling behavior expected from Eq. (7). All numerics in this Rapid Communication have been done using the algorithm explained in detail in [15]. The value of the dynamical scaling exponent a does not determine the transient spectral exponent x. One can attempt to determine this exponent by assuming that $F(\xi)\sim A\xi^{-x}$ as $\xi\to 0$ and trying to balance the leading terms in Eq. (4) with $a=2$,

$$(x-2)A\xi^{-x}=S_1[A\xi^{-x}]+S_2[A\xi^{-x}]+S_3[A\xi^{-x}]. \tag{8}$$

Using the scaling properties of the collision integrals together with the Zakharov transformations [1] we rewrite Eq. (8) in a relatively compact form

$$(x-2)A\xi^{-x}=A^2\xi^{\lambda-2x+1}I(x), \tag{9}$$

where (see [18] for details)

$$I(x)=\int K(\xi_1,\xi_2)(\xi_1\xi_2)^{-x}[1-\xi_1^x-\xi_2^x][1-\xi_1^{2x-\lambda-2} - \xi_2^{2x-\lambda-2}]\delta(1-\xi_1-\xi_2)d\xi_1 d\xi_2. \tag{10}$$

Note that the integral in Eq. (10) vanishes when $x=1$ (the thermodynamic exponent) or $x=\frac{\lambda+3}{2}$ (the Kolmogorov-Zakharov exponent).

Equation (9) leads to relations

$$x=\lambda+1, \tag{11}$$

$$A=\frac{\lambda-1}{I(\lambda+1)}. \tag{12}$$

These results require that the integral $I(\lambda+1)$ is convergent and does not vanish. Convergence depends on the asymptot-

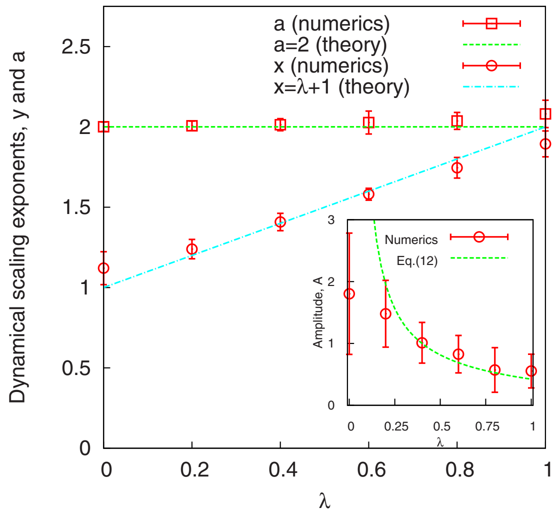

FIG. 2. (Color online) Dynamical scaling for $K(\omega_1,\omega_2)=(\omega_1\omega_2)^{\lambda/2}$. The main panel shows the dynamical exponents a and x. The inset shows the prefactor A. Error bars represent two standard deviations either side of the mean obtained by bootstrapping least-squares fits to random subsets of the numerical data.

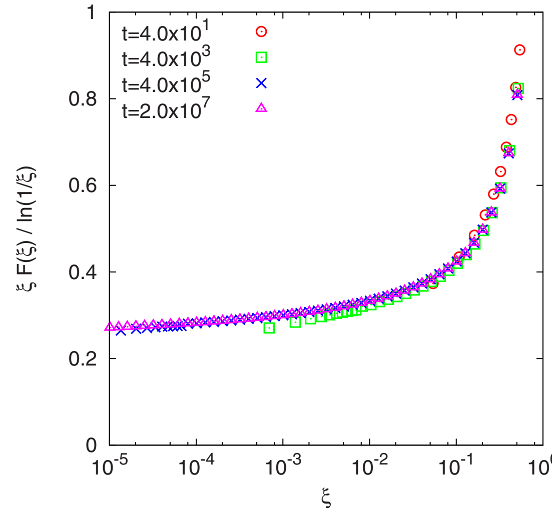

FIG. 3. (Color online) Scaling function $F(x)$ compensated for the theoretical small x divergence, $x^{-1}\ln(1/x)$, expected from Eq. (18).

ics of the kernel but is straightforward to check [18]. Figure 2 summarizes the results of a set of numerical simulations of Eq. (1) with the product kernel $K(\omega_1,\omega_2)=(\omega_1\omega_2)^{\lambda/2}$ for $0\le\lambda\le1$, for which there are no issues of convergence of the integral $I(\lambda+1)$. The theoretical value for the dynamical scaling exponent, $a=2$, is recovered in all cases. The measured values of the transient spectral exponent x and decay amplitude A agree well with Eqs. (11) and (12), except near $\lambda=0$.

The deviation from Eqs. (11) and (12) at $\lambda=0$ can be traced to the fact that the transient spectral exponent, $x=\lambda+1$, coincides with the thermodynamic equilibrium exponent, $x=1$, when $\lambda=0$. As a result, $I(x)$ vanishes and Eq. (9) fails to determine the asymptotics. It was remarked in [4] that this coincidence of exponents also occurs for forced optical turbulence in two dimensions. It was conjectured that, based on some analysis of the differential approximation to the collision integral, this should lead to a logarithmic correction to scaling. For the current problem, we can verify this explicitly for the case of decaying wave turbulence with constant kernel, the simplest example with $\lambda=0$.

For the constant kernel $K(\omega_1,\omega_2)=1$, Eq. (1) reads

$$\frac{dN_\omega}{dt}=\frac{1}{2}\sum_{\omega_1+\omega_2=\omega}N_{\omega_1}N_{\omega_2}-N_\omega\sum_{\omega_1\ge1}N_{\omega_1}-N_\omega\sum_{\omega_1<\omega}N_{\omega_1}+N_\omega\sum_{\omega_1>\omega}N_{\omega_1}+\sum_{\omega_1\ge1}N_{\omega_1}N_{\omega+\omega_1}. \tag{13}$$

The total wave action, $N(t)=\Sigma_{\omega\ge1}N_\omega(t)$, satisfies the equation [found by summing Eq. (13)]

$$\frac{dN}{dt}=-\frac{1}{2}\sum_{\omega\ge1}N_\omega^2. \tag{14}$$

The primary waves (monomers) evolve according to

$$\frac{dN_1}{dt}=-N_1^2+\sum_{\omega\ge1}N_\omega N_{\omega+1}. \tag{15}$$

If $F(\xi)$ diverges algebraically, as in Eq. (5), then substitution of Eq. (2) into Eq. (15) requires that $x-3=2x-4$ and $A=(2-x)\{1-\Sigma_{\omega\ge1}[\omega(\omega+1)]^{-x}\}^{-1}$. This would fix $x=1$ where it not for the fact that $\Sigma_{\omega\ge1}\{[\omega(\omega+1)]\}^{-1}=1$, so that the amplitude A diverges consistent with our earlier considerations. Let us instead assume that

$$F(\xi)\sim\xi^{-1}[\ln(1/\xi)]^{\rho}\quad\text{as}\quad\xi\to0, \tag{16}$$

where we have introduced a logarithmic exponent ρ, which may cancel this divergence. The tail of the wave spectrum then has the form

$$N_\omega(t)=\frac{A}{s(t)}\frac{1}{\omega}\left[\ln\left(\frac{s(t)}{\omega}\right)\right]^{\rho}\quad\text{for}\quad\omega\ll s(t). \tag{17}$$

Setting $k=1$ here gives the asymptotic form of $N_1(t)$. Substituting into Eq. (15) one finds that the leading term on the left-hand side is of order $s(t)^{-2}\ln[s(t)]^{\rho}$ and the leading-order term on the right-hand side is of order $s(t)^{-2}\ln[s(t)]^{2\rho-1}$ (*not* $s(t)^{-2}\ln[s(t)]^{2\rho}$ as one might naively expect owing to the cancellation alluded to above). Thus, we should choose $\rho=1$ so that the asymptotic form of $F(\xi)$ in the constant kernel case is

$$F(\xi)\sim\xi^{-1}\ln(1/\xi)\quad\text{as}\quad\xi\to0. \tag{18}$$

Thus, the apparently simplest model with constant kernel has a hidden degeneracy leading to nontrivial behavior. Figure 3 shows the numerical scaling function compensated according to Eq. (18). The plateau at small ξ supports Eq. (18). Determining A is easier using Eq. (14) for $N(t)$ since the sums which arise can be computed exactly. The total wave action is approximately

$$N(t) \approx As(t)^{-1}\sum_{\omega=1}^{s(t)} \omega^{-1} \ln\left[\frac{s(t)}{\omega}\right]$$
$$\approx As(t)^{-1}\int_{\omega=1}^{s(t)} \omega^{-1} \ln\left[\frac{s(t)}{\omega}\right] d\omega = \frac{A}{2}\frac{\ln[s(t)]^2}{s(t)}. \quad (19)$$

Substituting this into the left-hand side of Eq. (14), and Eq. (17) into the right-hand side, and computing the leading terms we find the balance

$$-\frac{As_0}{2}\frac{\ln[s(t)]^2}{s(t)^2} = -\frac{A^2}{2}\frac{\ln[s(t)]^2}{s(t)^2}\sum_{\omega=1}^{\infty}\frac{1}{\omega^2}.$$

The sum is $\pi^2/6$ from which $A/s_0=\pi^2/6$. Since $s(t)\sim s_0 t$ we have the following nontrivial asymptotic behaviors (validated numerically in Fig. 4) for the total density and number of primary waves:

$$N(t) \sim \frac{3}{\pi^2}\frac{(\ln t)^2}{t}, \quad N_1(t) \sim \frac{6}{\pi^2}\frac{\ln t}{t}. \quad (20)$$

To conclude, we have used the analogy between the isotropic three-wave turbulence and aggregation–fragmentation processes to study analytically the decay kinetics of three-wave turbulence in the infinite capacity regime. We showed that the transient exponent is $\lambda+1$, where λ is the degree of homogeneity of the wave interaction kernel, and derived a formula for the decay amplitude. When $\lambda=0$, the transient and thermodynamic equilibrium exponents coincide resulting in logarithmic corrections to scaling. For the constant kernel case, we computed this correction explicitly and calculated the corrected decay amplitude.

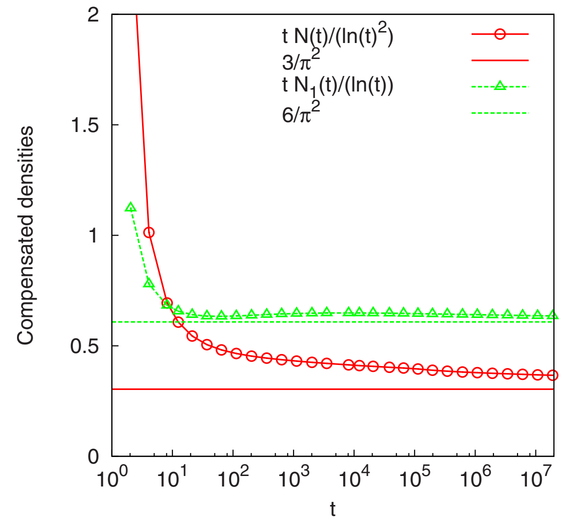

FIG. 4. (Color online) Time decay of total wave action, $N(t)$, and primary wave action, $N_1(t)$, compensated by the theoretically predicted asymptotic decay rates in Eq. (20). Theoretical plateau values are attained to within a few percent.

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